

# RECENT DEVELOPMENTS IN SPECTRAL SYNTHESIS FOR EXPONENTIAL SYSTEMS AND FOR NON-SELF-ADJOINT OPERATORS

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*Dedicated to Nikolai K. Nikolski on the occasion of his 70-th birthday*

**ABSTRACT.** We survey recent results concerning the hereditary completeness of some special systems of functions and the spectral synthesis problem for a related class of linear operators. We present a solution of the spectral synthesis problem for systems of exponentials in  $L^2(-\pi, \pi)$ . Analogous results are obtained for the systems of reproducing kernels in the de Branges spaces of entire functions. We also apply these results (via a functional model) to the spectral theory of rank one perturbations of compact self-adjoint operators.

## 1. INTRODUCTION

Spectral synthesis for linear operators and hereditary completeness of systems of vectors in a Hilbert space are among numerous mathematical interests of Nikolai Nikolski. His influence on the subject is enormous. The authors are deeply grateful to Nikolai Nikolski, who introduced them to the field of the spectral function theory and posed many of the problems studied in the paper.

The aim of the present paper is to give a short and accessible overview of our recent results on two closely related topics. The first one concerns the hereditary completeness for systems of exponentials in  $L^2(-\pi, \pi)$  or, more generally, for systems of reproducing kernels in the de Branges spaces of entire functions (studied by the first three authors in [2]). The second topic is the spectral theory and, in particular, the spectral synthesis problem for rank one perturbations of compact self-adjoint operators. This part is based on the preprints [3, 4] of the first and the fourth authors.

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**1.1. Hereditary completeness.** A system of vectors  $\{x_n\}_{n \in N}$  in a separable Hilbert space  $H$  is said to be *exact* if it is both *complete* (i.e.,  $\overline{\text{Span}}\{x_n\} = H$ ) and *minimal* (i.e.,  $\overline{\text{Span}}\{x_n\}_{n \neq n_0} \neq H$  for any  $n_0$ ). Given an exact system we consider its (unique) *biorthogonal* system  $\{\tilde{x}_n\}_{n \in N}$  which satisfies  $(x_m, \tilde{x}_n) = \delta_{mn}$ . We associate to every element  $x \in H$  its formal Fourier series

$$(1.1) \quad x \sim \sum_{n \in N} (x, \tilde{x}_n) x_n.$$

In what follows, we will also need the notion of a *linear summation method* for a vector series  $\sum_{n \geq 1} v_n$ , where  $v_n$  are vectors in  $H$ . Let  $R = (R_{jk})_{j,k=1}^\infty$  be a doubly infinite matrix, whose entries satisfy the consistency properties

$$\sup_{j \geq 1} \sum_{k=1}^{\infty} |R_{jk}| < \infty, \quad \lim_{j \rightarrow \infty} R_{jk} = 0, \quad k \geq 1, \quad \lim_{j \rightarrow \infty} \sum_{k=1}^{\infty} R_{jk} = 1.$$

The series  $\sum_{n \geq 1} v_n$  is summable to a vector  $v$  by the method  $R$  if the limit

$$\lim_{j \rightarrow \infty} \sum_{k=1}^{\infty} R_{jk} \sum_{n=1}^k v_n$$

exists and equals to  $v$ .

The system  $\{x_n\}_{n \in N}$  is said to be *hereditarily complete* if, for any  $x \in H$ ,

$$x \in \overline{\text{Span}}\{(x, \tilde{x}_n) x_n\},$$

which may be understood as a possibility of reconstructing (in a very weak sense) the vector  $x$  from its Fourier series. Clearly, if the series (1.1) admits a linear summation method, then the system  $\{x_n\}$  is hereditarily complete.

An equivalent definition of hereditary completeness is that for any partition  $N = N_1 \cup N_2$ ,  $N_1 \cap N_2 = \emptyset$ , the system

$$\{x_n\}_{n \in N_1} \cup \{\tilde{x}_n\}_{n \in N_2}$$

is complete in  $H$ . In particular, this requires that the biorthogonal system  $\{\tilde{x}_n\}$  be also complete (which is not always true). If both  $\{x_n\}$  and  $\{\tilde{x}_n\}$  are complete, but the system is not hereditarily complete, we say that  $\{x_n\}$  is *nonhereditarily complete*.

A general approach to constructing nonhereditarily complete systems was developed by Dovbysh, Nikolski and Sudakov [9, 10], where it was shown, in particular, that for any condition of closeness to an orthogonal basis (weaker than the quadratical closeness) there are uniformly minimal nonhereditarily complete systems that meet this condition. We recall that a complete minimal system that is quadratically close to an orthogonal basis is always a Riesz basis, by a classical theorem of Bari.

**1.2. Spectral synthesis of linear operators.** Let  $T$  be a bounded linear operator in a Hilbert space  $H$  which has a complete set of eigenvectors (or, more generally, a complete set of root vectors)  $\{x_n\}$ . The operator  $T$  is said to admit the *spectral synthesis* if for any  $T$ -invariant closed linear subspace  $M$  of  $H$  the restriction of  $T$  onto  $M$  has a complete set of root vectors. Equivalently,  $M = \overline{\text{Span}}\{x_n : x_n \in M\}$ , that is,  $M$  is generated by the root vectors it contains.

The spectral synthesis problem in this form was posed by Wermer [30] who showed that any unitary or self-adjoint operator with a complete set of eigenvectors admits the spectral synthesis as well as any compact normal operator. The first example of a compact operator with a complete set of eigenvectors (and with a trivial kernel) which does not admit the spectral synthesis goes back to Hamburger [15], who constructed a compact operator with a complete set of eigenvectors, whose restriction to an invariant subspace is a nonzero Volterra operator.

Further results about the spectral synthesis of compact operators were obtained by Nikolski [22] and Markus [20]. Let us mention the following theorem of Nikolski [22], which shows that any Volterra operator can be realized as a part of compact operator with a complete set of eigenvectors: *For any Volterra operator  $T$  in a Hilbert space  $H$  there exists a Hilbert space  $H'$  and a compact operator  $T' : H \oplus H' \rightarrow H \oplus H'$  such that  $T'H \subset H$ ,  $T'|_H = T$ , and the eigenfunctions of  $T'$  corresponding to nonzero eigenvalues are complete in  $H \oplus H'$ .* In this situation one can control the decay of the singular values of  $T'$  in terms of the corresponding information on  $T$ , see [28].

On the other hand, it was shown in [20] that if  $T$  is a compact operator with a complete set of eigenvectors  $\{x_n\}$ , then  $T$  admits the spectral synthesis if and only if  $\{x_n\}$  is hereditarily complete. For an extensive review of the subject see the surveys [24], [17, Chapter 4], and [23].

**1.3. Hereditary completeness for exponential systems.** It is natural to study the problem of hereditary completeness for special systems in functional spaces, e.g. those which appear as families of eigenvectors and root vectors of a certain operator. Exponential systems form an important class in this respect. Let  $\Lambda = \{\lambda_n\} \subset \mathbb{C}$  and let  $e_\lambda(t) = \exp(i\lambda t)$ . We consider the exponential system  $\{e_\lambda\}_{\lambda \in \Lambda}$  in  $L^2(-a, a)$ ,  $a > 0$ . By the result of Young [31], for any exact system of exponentials  $\{e_\lambda\}_{\lambda \in \Lambda}$  its biorthogonal system  $\{\tilde{e}_\lambda\}_{\lambda \in \Lambda}$  is complete.

Another important class of systems are families of the reproducing kernels in the so-called model subspaces  $K_\Theta = H^2 \ominus \Theta H^2$  of the Hardy space  $H^2$  (either in the disk or in the half plane), where  $\Theta$  is an inner function. The spaces  $K_\Theta$  are known especially

for their role in the Nagy–Foiaş functional model. They have numerous applications in operator-related function theory (we refer to Nikolski’s monographs [25, 26]).

The following two questions were posed by Nikolski:

*Does there exist an exact system of reproducing kernels in  $K_\Theta$ , such that its biorthogonal system is incomplete?*

*Does there exist an exact system of reproducing kernels in  $K_\Theta$  with the complete biorthogonal system, which is nonhereditarily complete?*

Both questions have positive answers. Examples of exact systems of reproducing kernels in  $K_\Theta$  with incomplete biorthogonal systems were constructed in [1] (see Theorem 3.1).

There is a unitary operator (constructed from the Fourier transform), that sends any exponential system in  $L^2(-a, a)$  to a system of reproducing kernels in the model space  $K_{e^{2iaz}}$  in the upper half-plane. Therefore the hereditary completeness of exponential systems is a special case of the second question. One of the main results of [2] is that there exist nonhereditarily complete systems of exponentials.

**Theorem 1.1.** *There exists a system of exponentials  $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ ,  $\Lambda \subset \mathbb{R}$ , which is complete and minimal in  $L^2(-\pi, \pi)$ , but is not hereditarily complete.*

Thus, in general there exists no linear summation method for nonharmonic Fourier series.

**Corollary 1.2.** *There exist a system of exponentials  $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ ,  $\Lambda \subset \mathbb{R}$ , which is complete and minimal in  $L^2(-\pi, \pi)$ , such that the corresponding Fourier series  $\sum_{\lambda \in \Lambda} (f, \tilde{e}_\lambda) e_\lambda$  admit no linear summation method.*

However, surprisingly, the exponential systems are hereditarily complete up to a one-dimensional defect (see Theorem 2.1 and the discussion in the next section). Analogous results are obtained for systems of reproducing kernels in the de Branges spaces of entire functions (which correspond to model spaces generated by inner functions  $\Theta$  meromorphic in  $\mathbb{C}$ ), see Section 3.

**1.4. One-dimensional perturbations of compact self-adjoint operators.** Systems of reproducing kernels in Hilbert spaces of analytic functions may serve as models for abstract systems of vectors in a general Hilbert space. As was pointed out by Nikolski, the model subspaces form a promising setting in this respect due to their rich structure.

One may ask, whether the exact systems of exponentials or reproducing kernels of de Branges spaces may be interpreted as eigenfunctions of bounded linear operators from some special class. Here the rank one perturbations of bounded self-adjoint operators come into play.

Let  $A^\circ$  be a compact self-adjoint operator on a Hilbert space  $H$  with simple spectrum and with a trivial kernel. Let  $a^\circ, b^\circ \in H$  and let  $L^\circ = L(A^\circ, a^\circ, b^\circ)$  be its one-dimensional perturbation<sup>1</sup>, that is,

$$L^\circ f = A^\circ f + (f, b^\circ)a^\circ, \quad f \in H.$$

Though this is one of the simplest and most natural classes of compact linear operators, it seems that our knowledge of the spectral structure of these operators is very incomplete.

One of the basic questions addressed in [3] was to understand when the perturbed operator  $L^\circ$  has a complete set of eigenfunctions. Notice that the classical theory covers only the case of so-called weak perturbations, i.e., perturbations of the form  $L^\circ = A^\circ(I + T^\circ)$  where  $T^\circ$  is a compact operator (theorems of Keldysh and Matsaev), as well as some other special classes of operators (e.g., dissipative ones), see [21] and [12, Chapter V].

In our study, we use a functional model for rank-one perturbations of self-adjoint operators, which is close to a model by Kapustin given in [16]. See Section 4 below for more details. This model allows us to obtain new positive results on completeness and spectral synthesis for rank one perturbations of compact self-adjoint operators as well as a number of rather unexpected counterexamples. These counterexamples show how rich and complicated the spectral structure becomes even for rank one perturbations as soon as we leave the area of dissipative operators.

Here we mention one of these examples (for the precise statement see Theorem 5.6):

**Theorem 1.3.** *Under some mild separation conditions on the spectrum of  $A^\circ$  there exists a rank one perturbation  $L^\circ$  of  $A^\circ$  which does not admit the spectral synthesis.*

The paper is organized as follows. In Section 2 we discuss the solution of the spectral synthesis problem for exponential systems, while in Section 3 some extensions to the general de Branges spaces are considered. The functional model for rank one perturbations and its corollaries are discussed in Sections 4 and 5. We conclude the article with a list of open problems.

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<sup>1</sup>We use the notation  $L^\circ, A^\circ$  etc. to distinguish the compact operators from the case of *singular* rank one perturbations of *unbounded* self-adjoint operators, which will be respectively denoted by  $L$  and  $A$ .

## 2. HEREDITARY COMPLETENESS FOR EXPONENTIAL SYSTEMS

As usual, applying the Fourier transform  $\mathcal{F}$  one reduces the problem for exponential systems in  $L^2(-\pi, \pi)$  to the same problem for systems of reproducing kernels in the Paley–Wiener space  $\mathcal{PW}_\pi = \mathcal{FL}^2(-\pi, \pi)$ . Recall that the reproducing kernel of  $\mathcal{PW}_\pi$  corresponding to a point  $\lambda \in \mathbb{C}$  is of the form

$$K_\lambda(z) = \frac{\sin \pi(z - \bar{\lambda})}{\pi(z - \bar{\lambda})}, \quad f(\lambda) = (f, K_\lambda)_{\mathcal{PW}_\pi}.$$

In [2], we solve the problem of hereditary completeness for exponential systems. Namely, we show that the hereditary completeness holds up to a possible one-dimensional defect.

Let  $\Lambda \subset \mathbb{C}$  be such that the system of reproducing kernels  $\{K_\lambda\}_{\lambda \in \Lambda}$  is exact in the Paley–Wiener space  $\mathcal{PW}_\pi$ . Then the biorthogonal system  $\{g_\lambda\}$  is given by

$$g_\lambda(z) = \frac{G(z)}{G'(\lambda)(z - \lambda)},$$

where  $G$  is the so-called *generating function* of the set  $\Lambda$ : the entire function  $G$  has only simple zeros at the points of  $\Lambda$  and  $g_\lambda \in \mathcal{PW}_\pi$ ,  $\lambda \in \Lambda$ . By the above-mentioned result of Young,  $\{g_\lambda\}$  is also an exact system. It is well known that  $G$  is of exponential type  $\pi$ .

The following theorem gives a complete answer to the hereditary completeness of reproducing kernels (equivalently exponential systems).

**Theorem 2.1** ([2], Theorems 1.1, 1.3). *1. If  $\{K_\lambda\}_{\lambda \in \Lambda}$  is exact in the Paley–Wiener space  $\mathcal{PW}_\pi$ , then for any partition  $\Lambda = \Lambda_1 \cup \Lambda_2$ , the orthogonal complement in  $\mathcal{PW}_\pi$  to the system*

$$(2.1) \quad \{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$$

*is at most one-dimensional.*

*2. There exists a system  $\{K_\lambda\}_{\lambda \in \Lambda}$ ,  $\Lambda \subset \mathbb{R}$ , which is exact in  $\mathcal{PW}_\pi$ , but is not hereditarily complete (thus, there exists a partition  $\Lambda = \Lambda_1 \cup \Lambda_2$ , such that the orthogonal complement in  $\mathcal{PW}_\pi$  to the system (2.1) is one-dimensional).*

Thus, hereditary completeness may fail even for exponential systems (reproducing kernels of the Paley–Wiener space), which answers the question of Nikolski. Further counterexamples will be discussed in the next section.

The set of the exceptional partitions  $\Lambda = \Lambda_1 \cup \Lambda_2$  (for which the orthogonal complement to (2.1) is nontrivial) is in a sense very small. This nontrivial defect cannot appear unless the sequence  $\Lambda_1$  has zero density. Given a sequence  $\Lambda$ , set

$$D_+(\Lambda) = \limsup_{r \rightarrow \infty} \frac{n_r(\Lambda)}{2r},$$

where  $n_r(\Lambda)$  is the usual counting function of the sequence  $\Lambda$ ,  $n_r(\Lambda) = \#\{\lambda \in \Lambda, |\lambda| \leq r\}$ .

**Theorem 2.2** ([2], Theorem 1.2). *Let  $\Lambda \subset \mathbb{C}$ , let the system  $\{K_\lambda\}_{\lambda \in \Lambda}$  be exact in  $\mathcal{PW}_\pi$ , and let the partition  $\Lambda = \Lambda_1 \cup \Lambda_2$  satisfy  $D_+(\Lambda_1) > 0$ . Then the system (2.1) is complete in  $\mathcal{PW}_\pi$ .*

Theorem 2.2 shows that there is a strong asymmetry between the systems of reproducing kernels and their biorthogonal. The completeness of (2.1) may fail only when we take a sparse (but infinite!) subsequence  $\Lambda_1$ .

### 3. SYSTEMS OF REPRODUCING KERNELS IN THE DE BRANGES SPACES

**3.1. Preliminaries on de Branges spaces.** Let  $E$  be an entire function in the Hermite–Biehler class, that is  $E$  has no zeros in  $\mathbb{C}_+ \cup \mathbb{R}$ , and

$$|E(z)| > |E^*(z)|, \quad z \in \mathbb{C}_+,$$

where  $E^*(z) = \overline{E(\bar{z})}$ . With any such function we associate the *de Branges space*  $\mathcal{H}(E)$  which consists of all entire functions  $F$  such that  $F/E$  and  $F^*/E$  restricted to  $\mathbb{C}_+$  belong to the Hardy space  $H^2 = H^2(\mathbb{C}_+)$ . The inner product in  $\mathcal{H}(E)$  is given by

$$(F, G)_E = \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2} dt.$$

The reproducing kernel of the de Branges space  $\mathcal{H}(E)$  corresponding to the point  $w \in \mathbb{C}$  is given by

$$K_w(z) = \frac{\overline{E(w)}E(z) - \overline{E^*(w)}E^*(z)}{2\pi i(\bar{w} - z)} = \frac{\overline{A_E(w)}B_E(z) - \overline{B_E(w)}A_E(z)}{\pi(z - \bar{w})},$$

where we use the standard decomposition  $E = A_E - iB_E$ ,  $A_E = \frac{E+E^*}{2}$ ,  $B_E = \frac{E^*-E}{2i}$ .

The Hilbert spaces of entire functions  $\mathcal{H}(E)$  were introduced by L. de Branges [7] in relation to the inverse spectral problems for differential operators. These spaces are also of a great interest from the function theory point of view. The Paley–Wiener space  $\mathcal{PW}_a$  is the de Branges space corresponding to  $E(z) = \exp(-iaz)$ . Also, de Branges spaces are canonically isomorphic to model spaces generated by meromorphic inner functions: if  $\Theta$  has a meromorphic continuation to the whole plane, then  $\Theta = E^*/E$  for a function  $E$  in the Hermite–Biehler class and the mapping  $f \mapsto Ef$  is a unitary operator from  $K_\Theta$  onto  $\mathcal{H}(E)$ , which maps reproducing kernels onto reproducing kernels.

An important characteristics of the de Branges space  $\mathcal{H}(E)$  is its phase function, that is, an increasing  $C^\infty$ -function  $\varphi$  such that  $E(t)\exp i\varphi(t) \in \mathbb{R}$ ,  $t \in \mathbb{R}$  (thus, essentially,  $\varphi = -\arg E$  on  $\mathbb{R}$ ). Clearly, for  $\mathcal{H}(E) = \mathcal{PW}_a$ ,  $\varphi(t) = at$ . If  $\varphi' \in L^\infty(\mathbb{R})$  (in which



case we say that  $\varphi$  has sublinear growth), the space  $\mathcal{H}(E)$  shares some properties with the Paley–Wiener spaces.

A crucial property of the de Branges spaces is the existence of orthogonal bases of reproducing kernels corresponding to real points [7]. For  $\alpha \in [0, \pi)$  we consider the set of points  $t_n \in \mathbb{R}$  such that

$$(3.1) \quad \varphi(t_n) = \alpha + \pi n, \quad n \in \mathbb{Z}.$$

Thus,  $\{t_n\}$  is the zero set of the function  $e^{i\alpha}E - e^{-i\alpha}E^*$ .

If the points  $t_n$  are defined by (3.1), then the system of reproducing kernels  $\{K_{t_n}\}$  is an orthogonal basis for  $\mathcal{H}(E)$  for each  $\alpha \in [0, \pi)$  except, may be, one ( $\alpha$  is an exceptional value if and only if  $e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$ ). In particular, if  $A_E \notin \mathcal{H}(E)$ , then  $\{\frac{A_E}{z-t_n}\}$  is the orthogonal basis for  $\mathcal{H}(E)$  corresponding to  $\alpha = \pi/2$ .

It should be mentioned that the index set for the sequence  $\{t_n\}$  is either  $\mathbb{Z}$  or  $\mathbb{Z}_\pm$ ; in the latter case we may need to add  $k\pi$  to  $\varphi$ , for some  $k \in \mathbb{Z}$ .

The norm of  $K_t$ ,  $t \in \mathbb{R}$ , is given by  $\|K_t\|_E^2 = \pi^{-1}|E(t)|^2\varphi'(t)$ . Thus, if we put  $\mu = \sum_n \|K_{t_n}\|^{-2}\delta_{t_n}$ , then the embedding  $\mathcal{H}(E) \rightarrow L^2(\mu)$  is a unitary operator. One should think of the sequence  $\{t_n\}$  as of a spectral characteristics of the space  $\mathcal{H}(E)$ .

### 3.2. Hereditary completeness for reproducing kernels of the de Branges spaces.

As in the Paley–Wiener case we denote by  $G$  the generating function of the set  $\Lambda$ , and the system biorthogonal to the system  $\{K_\lambda\}_{\lambda \in \Lambda}$  is given by  $\{g_\lambda\}_{\lambda \in \Lambda}$ , where  $g_\lambda(z) = \frac{G(z)}{G'(\lambda)(z-\lambda)}$ .

We say that  $\varphi$  is of *tempered growth* if  $\varphi'(t) = O(|t|^N)$ ,  $|t| \rightarrow \infty$ , for some  $N$ .

The completeness of the systems biorthogonal to exact systems of reproducing kernels was studied in [1, 11]. In particular, it was shown in [11] that such biorthogonal systems are always complete when  $\varphi' \in L^\infty(\mathbb{R})$ .

The following result is obtained in [1]:

**Theorem 3.1** ([1], Theorems 1.1, 1.2). *1. If  $e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$  for some  $\alpha \in [0, \pi)$ , then there exists an exact system of reproducing kernels, whose biorthogonal is incomplete.*

*2. If  $\varphi$  is of tempered growth, then any system  $\{g_\lambda\}_{\lambda \in \Lambda}$  biorthogonal to an exact system of reproducing kernels has at most a finite-dimensional defect in  $\mathcal{H}(E)$ . If, moreover,  $e^{i\alpha}E - e^{-i\alpha}E^* \notin \mathcal{H}(E)$  for any  $\alpha \in [0, \pi)$ , then  $\{g_\lambda\}_{\lambda \in \Lambda}$  is complete in  $\mathcal{H}(E)$ .*

The method of the proof of Theorem 2.1 extends to the case of the de Branges spaces with sublinear or tempered growth of the phase.

**Theorem 3.2** ([2], Theorem 1.4, Theorem 5.3). *Let  $\mathcal{H}(E)$  be a de Branges space such that  $\varphi$  is of tempered growth. If the system of reproducing kernels  $\{K_\lambda\}_{\lambda \in \Lambda}$  is exact in  $\mathcal{H}(E)$ ,*



then for any partition  $\Lambda = \Lambda_1 \cup \Lambda_2$ , the orthogonal complement in  $\mathcal{H}(E)$  to the system

$$(3.2) \quad \{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$$

is of finite dimension.

If, moreover,  $\varphi' \in L^\infty(\mathbb{R})$ , then the orthogonal complement in  $\mathcal{H}(E)$  to the system (3.2) is at most one-dimensional.

A crucial step in the proofs of Theorems 2.1 and 3.2 is the use of expansions of functions in  $\mathcal{PW}_\pi$  or in  $\mathcal{H}(E)$  with respect to *two different* orthogonal bases of reproducing kernels. At first glance it may look like an artificial trick; however it should be noted that the existence of two orthogonal bases of reproducing kernels is a property which characterizes de Branges spaces among all Hilbert spaces of entire functions (see [5, 6]). Therefore, we believe this method to be intrinsically related to the deep and complicated geometry of de Branges spaces.

We show that nonhereditary completeness for reproducing kernels is possible in many de Branges spaces. Namely, we construct such examples under some mild restrictions on the spectrum  $\{t_n\}$  (in particular, all power growth spectra  $|t_n| = |n|^\gamma$ ,  $\gamma > 0$ ,  $n \in \mathbb{N}$  or  $n \in \mathbb{Z}$ , satisfy these restrictions).

Here and later on we use the notation  $U(t) \lesssim V(t)$  if there is a constant  $C > 0$  such that  $U(t) \leq CV(t)$  holds for all  $t$  in the set in question.

**Theorem 3.3** ([2], Theorem 1.6). *Let  $\{t_n\}$  be a sequence of real points such that  $t_n < t_{n+1}$  and  $|t_n| \rightarrow \infty$ ,  $n \rightarrow \infty$ . Assume that, for some  $N > 0$ ,*

$$(3.3) \quad |t_n|^{-N} \lesssim t_{n+1} - t_n = o(|t_n|), \quad |n| \rightarrow \infty.$$

*Then there exists a de Branges space  $\mathcal{H}(E)$  such that  $\varphi$  is of tempered growth,  $\{t_n\}$  is the zero set of the function  $A_E \notin \mathcal{H}(E)$  and there is an exact system of reproducing kernels  $\{K_\lambda\}$  in  $\mathcal{H}(E)$  such that its biorthogonal system is complete, but the original system  $\{K_\lambda\}$  is nonhereditarily complete.*

For a question on the size of the orthogonal complement to the systems (3.2) see Problem 5 in Section 6 below.

The statement of Theorem 3.3 is in sharp contrast with the fact that there exist de Branges spaces where *any exact system of reproducing kernels is hereditarily complete*. The reason for that is, however, that a de Branges space is (in essence) uniquely defined by the spectrum  $\{t_n\}$  and the masses  $\mu_n = 1/\varphi'(t_n)$  of the spectral measure, while given only a spectrum there is a lot of freedom in prescribing  $\mu_n$ . Note also that in the example below condition (3.3) is not satisfied.

**Example 3.4.** Let  $\{t_n\}_{n \in \mathbb{N}}$  be a sequence such that  $t_n > 0$  and  $\inf_n t_{n+1}/t_n > 1$ , and let  $\mu_n \equiv t_n^\gamma$ , where  $0 < \gamma < 2$ . Denote by  $A_E$  the canonical product with the zeros  $t_n$  and define the entire function  $B_E$  by

$$\frac{B_E(z)}{A_E(z)} = \sum_n \mu_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right).$$

Then the function  $E = A_E - iB_E$  is in the Hermite–Biehler class. Making use of the results of [2, Section 5] it is easy to show that any exact system of reproducing kernels in  $\mathcal{H}(E)$  is hereditarily complete. Otherwise, by the results of [2], there would exist a function  $h \in \mathcal{H}(E)$ ,  $h \neq 0$ , orthogonal to a system (3.2) such that for a sequence  $\{a_n\} \in \ell^2$  the functions

$$\frac{h(z)}{A_E(z)} = \sum_n \frac{\bar{a}_n \mu_n^{1/2}}{z - t_n} \quad \text{and} \quad \sum_n \frac{|a_n|^2}{z - t_n}$$

have infinitely many common zeros on the real line. However, comparing the asymptotic of the distances from these common zeros to  $\{t_n\}$ , one can show that the set of common zeros can not be infinite.

Let us also mention here a result by Vasyunin [29] (see also [25, Chapter VIII]) characterizing the Abel summability of the Cauchy kernels  $\{(z - \bar{\lambda})^{-1} : B(\lambda) = 0\}$  expansions in the model spaces  $K_B$ .

#### 4. FUNCTIONAL MODEL FOR SINGULAR RANK ONE PERTURBATIONS OF UNBOUNDED SELF-ADJOINT OPERATORS

To apply the technique of the entire functions theory, it is more convenient to work not with perturbations of compact self-adjoint operators, but with their unbounded inverses. These operators can be understood as certain *singular* rank one perturbations of self-adjoint operators.

Let  $\{t_n\}$  be an increasing sequence of real numbers (where  $n \in \mathbb{N}$  or  $n \in \mathbb{Z}$ ) such that  $|t_n| \rightarrow \infty$  as  $|n| \rightarrow \infty$ , and let  $\mu = \sum_n \mu_n \delta_{t_n}$ . In what follows  $A$  is always the operator of multiplication by the independent variable  $x$  in  $L^2(\mu)$  (thus,  $A$  is a self-adjoint operator with simple discrete spectrum). Moreover, we assume that  $0 \notin \{t_n\}$ , and so  $A^{-1}$  is a bounded operator in  $L^2(\mu)$ .

By a *singular rank one perturbation* of  $A$  we mean the following operator. Let  $a, b$  be two functions such that

$$\frac{a}{x}, \frac{b}{x} \in L^2(\mu)$$

(however, possibly,  $a, b \notin L^2(\mu)$ ). We write  $a(t_n) = a_n$ ,  $b(t_n) = b_n$ . Let  $\varkappa \in \mathbb{C} \setminus \{0\}$  be a constant such that

$$(4.1) \quad \varkappa \neq \int_{\mathbb{R}} x^{-1} a(x) \bar{b}(x) d\mu(x) \quad \text{in the case when } a(x) \in L^2(\mu).$$

We associate to any such data  $(a, b, \varkappa)$  a linear operator  $L = L(A, a, b, \varkappa)$ , defined as follows:

$$\begin{aligned} \mathcal{D}(L) &:= \{y = y_0 + c \cdot A^{-1}a : \\ &\quad c \in \mathbb{C}, y_0 \in \mathcal{D}(A), \varkappa c + \langle y_0, b \rangle = 0\}; \\ Ly &:= Ay_0, \quad y \in \mathcal{D}(L). \end{aligned}$$

Condition (4.1) is equivalent to the uniqueness of the decomposition  $y = y_0 + c \cdot A^{-1}a$  in the above formula for  $\mathcal{D}(L)$ , and so  $L$  is correctly defined. Note also that  $L = L(A, a, b, \varkappa)$  is densely defined if and only if the singular perturbation  $L(A, b, a, \overline{\varkappa})$  is correctly defined. In this case  $L^*$  is correctly defined and  $L^* = L(A, b, a, \overline{\varkappa})$ .

This construction can be extended to what can be called singular rank  $n$  perturbations of an unbounded operator  $A$ , see [3]. The following fact [3] motivates these definitions: if  $A$  and  $L$  are ordinary differential operators, defined by the same regular differential expression of order  $n$  and different abstract boundary relations ( $n$  independent relations in both cases), then  $L$  is a singular perturbation of  $A$  of rank less or equal to  $n$ .

Singular rank one perturbations defined above are essentially (unbounded) algebraic inverses to bounded rank one perturbations of compact self-adjoint operators. If the triplet  $(a, b, \varkappa)$  satisfies (4.1), then the bounded operator  $A^{-1} - \varkappa^{-1}A^{-1}a(A^{-1}b)^*$  has a trivial kernel, and

$$(4.2) \quad L(A, a, b, \varkappa) = (A^{-1} - \varkappa^{-1}A^{-1}a(A^{-1}b)^*)^{-1}.$$

Here we denote by  $A^{-1}a(A^{-1}b)^*$  the bounded rank one operator  $A^{-1}a(A^{-1}b)^*f = (f, A^{-1}b)A^{-1}a$ ,  $f \in L^2(\mu)$ . Conversely, if  $A^\circ$  is a compact self-adjoint operator with a trivial kernel and  $L^\circ = A^\circ + a^\circ(b^\circ)^*$  is its rank one perturbation such that  $\text{Ker } L^\circ = 0$ , then the algebraic inverse  $(L^\circ)^{-1}$  is a singular rank one perturbation of  $(A^\circ)^{-1}$ .

The following functional model for singular rank one perturbations was obtained in [3]. It is closely related to the model for rank one perturbations of singular unitary operators due to Kapustin [16] (a more general model can be found in [27]).

**Theorem 4.1** ([3], Theorem 0.7). *Let  $L = L(A, a, b, \varkappa)$  be a singular rank one perturbation of  $A$ , where  $b$  is a cyclic vector for the resolvent of  $A$ , i.e.,  $b_n \neq 0$  for any  $n$ . Then there*

exists a de Branges space  $\mathcal{H}(E)$  and an entire function  $G$  such that  $E + E^* \notin \mathcal{H}(E)$

$$(4.3) \quad G \notin \mathcal{H}(E), \quad \frac{G(z)}{z - z_0} \in \mathcal{H}(E) \quad \text{if} \quad G(z_0) = 0,$$

and  $L$  is unitary equivalent to the operator  $T = T(E, G)$  which acts on  $\mathcal{H}(E)$  by the formulas

$$\mathcal{D}(T) := \{F \in \mathcal{H}(E) : \text{there exists } c = c(F) \in \mathbb{C} : zF - cG \in \mathcal{H}(E)\},$$

$$TF := zF - cG, \quad F \in \mathcal{D}(T).$$

Conversely, any pair  $(E, G)$  where  $E$  is an Hermite–Biehler function such that  $A_E \notin \mathcal{H}(E)$  and  $A_E(0) \neq 0$ , while the entire function  $G$  satisfies (4.3), corresponds to some singular rank one perturbation  $L = L(A, a, b, \varkappa)$  of the multiplication operator  $A$  in some space  $L^2(\mu)$  with  $x^{-1}a(x), x^{-1}b(x) \in L^2(\mu)$ .

In fact, in [3] a much more general setting is considered where  $\mu$  is an arbitrary Borel measure singular with respect to Lebesgue measure.

The functions  $E = A_E - iB_E$  and  $G$  appearing in the model for  $L(A, a, b, \varkappa)$  are related to the data  $(a, b, \varkappa)$  by the following formulas:

$$(4.4) \quad \frac{B_E(z)}{A_E(z)} = \delta + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n,$$

where  $\delta$  is an arbitrary real constant, and

$$(4.5) \quad \frac{G(z)}{A_E(z)} = \varkappa + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) a_n \overline{b_n} \mu_n,$$

Note, in particular, that  $A_E$  vanishes exactly on the set  $\{t_n\}$ . The model essentially uses the expansions with respect to the orthogonal basis  $\left\{ \frac{A_E}{z - t_n} \right\}$  of normalized reproducing kernels (or, in the case of general model spaces, the representations involving Clark measures).

If we denote by  $Z_G$  the zero set of  $G$ , then it is clear that the eigenfunctions of  $T$  are exactly the functions  $\frac{G(z)}{z - \lambda}$ ,  $\lambda \in Z_G$ . If the operator  $T^*$  (equivalently,  $L^*$ ) is well-defined, then the eigenfunctions of  $T^*$  are the functions  $\{K_\lambda\}_{\lambda \in Z_G}$ . We have the following corollary.

**Corollary 4.2** ([3], Theorem 2.5). *Let  $\mathcal{H}(E)$  be a de Branges space such that  $e^{i\alpha}E - e^{-i\alpha}E^* \notin \mathcal{H}(E)$  for any  $\alpha \in \mathbb{R}$ , and let  $\{t_n\}$  be the zero set of  $A_E$ ,  $t_n \neq 0$ . Put  $s_n = t_n^{-1}$  and let  $A^\circ$  be a compact self-adjoint operator with simple eigenvalues  $\{s_n\}$  and with a trivial kernel. Let  $\{K_\lambda\}_{\lambda \in \Lambda}$  be any exact system of reproducing kernels in  $\mathcal{H}(E)$ . Then there exists a bounded rank-one perturbation  $L^\circ$  of  $A^\circ$  such that*

$$\sigma(L^\circ) = \{\lambda^{-1} : \lambda \in \Lambda\},$$

$\ker L^\circ = \ker (L^\circ)^* = 0$ , and there is a unitary transform  $U : \mathcal{H}(E) \rightarrow H$ , which maps the system  $\{K_\lambda\}_{\lambda \in \Lambda}$  to a system of eigenvectors of  $L^\circ$ :  $L^\circ(UK_\lambda) = \lambda^{-1}(UK_\lambda)$ .

It follows that at the same time,  $U^*$  takes the biorthogonal system to  $\{K_\lambda\}_{\lambda \in \Lambda}$  into a system of eigenvectors of the adjoint operator  $(L^\circ)^*$ .

## 5. APPLICATIONS OF THE FUNCTIONAL MODEL

Using the above functional model we obtain a number of results concerning the following natural questions:

- (i) When does  $L = L(A, a, b, \varkappa)$  have a complete system of eigenvectors (i.e.,  $L$  is complete)?
- (ii) When does the completeness of  $L$  imply the completeness of  $L^*$ ?
- (iii) When does  $L$  admit the spectral synthesis?
- (iv) For which  $A$  does there exist a rank one perturbation  $L(A, a, b, \varkappa)$  with the spectrum at infinity (i.e.,  $L$  is the inverse to a Volterra operator)?

**5.1. Completeness of rank one perturbations of self-adjoint operators.** We say that  $L(A, a, b, \varkappa)$  is a *generalized weak perturbation*, if

$$(5.1) \quad \sum_n \frac{|a_n b_n| \mu_n}{|t_n|} < \infty, \quad \sum_n \frac{a_n \bar{b}_n \mu_n}{t_n} \neq \varkappa.$$

It is essentially a corollary of Matsaev's theorem [21] that any generalized weak perturbation is complete. We get another sufficient condition for completeness of  $L$  if we assume a certain positivity condition.

**Theorem 5.1** ([3], Theorems 0.1, 3.3). *1. If  $L = L(A, a, b, \varkappa)$  is a generalized weak perturbation, then  $L^*$  is correctly defined, and  $L$  and  $L^*$  are complete.*

*2. Suppose that  $a_n \bar{b}_n \geq 0$  for all but a finite number of values of  $n$  and  $\sum_n |t_n|^{-1} |a_n b_n| \mu_n = \infty$ . Then  $L^*$  is correctly defined, and  $L$  and  $L^*$  are complete.*

A typical example when (5.1) is satisfied is that there is  $\alpha \in [0, 1]$  such that

$$a \in |x|^\alpha L^2(\mu), \quad b \in |x|^{1-\alpha} L^2(\mu).$$

At the same time it is easy to show that even for rank one perturbations, when we relax slightly the generalized weakness property (5.1), the resulting perturbation may become the inverse to a Volterra operator. We state the corresponding result for perturbations of compact operators.

**Theorem 5.2** ([3], Theorem 0.6). *There exists a sequence  $s_n \rightarrow 0$  and a measure  $\mu = \sum_n \mu_n \delta_{s_n}$  with the following property: for any  $\alpha_1, \alpha_2 \geq 0$  with  $\alpha_1 + \alpha_2 < 1$  there exist  $a^\circ \in |x|^{\alpha_1} L^2(\mu)$  and  $b^\circ \in |x|^{\alpha_2} L^2(\mu)$  such that the perturbed operator  $A^\circ + a^\circ(b^\circ)^*$  (where  $A^\circ$  is the operator of multiplication by  $x$  in  $L^2(\mu)$ ) is a Volterra operator.*

Positive results on the completeness can also be obtained for rank one perturbations of  $A$  that are not generalized weak perturbations, under the assumption that the spectrum of  $A$  is *exponentially sparse*, which means that  $s_{n+1}/s_n < \gamma$  for all  $n > 0$  (with a similar condition for  $n < 0$ ) and for some small (absolute) constant  $\gamma \in (0, 1)$ .

**5.2. Relations between completeness of the perturbed operator and of its adjoint.** If a bounded operator  $T$  is complete, a trivial obstacle for completeness of  $T^*$  is that  $T$  may have a nontrivial kernel, while  $\ker T^* = 0$ . The first (highly nontrivial) examples of the situation where a compact operator  $T$  is complete and  $\ker T = 0$ , while  $T^*$  is not complete, were constructed by Hamburger [15]. In [8] Deckard, Foiaş, and Pearcy gave a simpler construction. By definition, in these examples, the eigenvector system of  $T$  cannot be hereditarily complete, because if it were true, the biorthogonal to this system (which are just the eigenvectors of  $T^*$ ) would be also complete.

In the above-mentioned examples one cannot conclude that the corresponding operator is a small (for instance, finite rank or trace class) perturbation of a self-adjoint operator. Corollary 5.5 below shows that one can find such examples among rank one perturbations of compact self-adjoint operators with almost arbitrary spectrum.

We start with the situation of singular rank one perturbations of unbounded operators. As follows from the functional model, completeness of  $L$  and  $L^*$  is reduced to the completeness of a system of reproducing kernels and of its biorthogonal system. The relations between these two completeness problems were studied in [1].

The next theorem shows that under certain additional assumptions the completeness of a singular rank one perturbation  $L$  implies the completeness of its adjoint.

**Theorem 5.3** ([3], Theorem 0.2). *Let the data  $(a, b, \varkappa)$  satisfy the property  $a \notin L^2(\mu)$ , and let the perturbation  $L = L(A, a, b, \varkappa)$  be complete. Assume that its adjoint  $L^*$  is correctly defined. Then  $L^*$  is also complete if any of the following conditions is fulfilled:*

- (i)  $|a_n|^2 \mu_n \lesssim |t_n|^{-N}$  for some  $N > 0$ ;
- (ii)  $|b_n a_n^{-1}| \lesssim |t_n|^{-N}$  for some  $N > 0$ .

In general, the situation is much subtler. Applying the results and methods from [1] we are able to give examples when the adjoint to a complete perturbation fails to be complete.

**Theorem 5.4** ([3], Theorem 0.3). *For any cyclic self-adjoint operator  $A$  with discrete spectrum, there exists a singular rank one perturbation  $L$  of  $A$  with real spectrum, which is not complete, while its adjoint  $L^*$  is correctly defined, has trivial kernel and is complete. Moreover, the orthogonal complement to the space spanned by the eigenvectors of  $L$  may be infinite-dimensional.*

**Corollary 5.5** ([3], Corollary 0.4). *For any compact self-adjoint operator  $A^\circ$  with simple spectrum  $\{s_n\}$ ,  $s_n \neq 0$ , there exists a bounded rank one perturbation  $L^\circ$  of  $A^\circ$  with real spectrum such that  $L^\circ$  is complete and  $\ker L^\circ = 0$ , while  $(L^\circ)^*$  is not complete.*

**5.3. Spectral synthesis for rank one perturbations.** In view of the theorem of Markus [20, Theorem 4.1] and Corollary 4.2, the results of Section 3 may be interpreted as the results about the spectral synthesis for rank one perturbations of compact self-adjoint operators. In particular, we give now the accurate statement for Theorem 1.3.

**Theorem 5.6** ([3], Theorem 0.5). *Let  $\{s_n\}$  be a sequence of real numbers (ordered so that  $s_n > 0$  and  $s_n$  decrease for  $n \geq 0$ , and  $s_n < 0$  and increase for  $n < 0$ ) and assume that for some  $N > 0$*

$$|s_n|^N \lesssim |s_{n+1} - s_n| = o(|s_n|), \quad |n| \rightarrow \infty.$$

*Let  $A^\circ$  be a compact self-adjoint operator with simple eigenvalues  $\{s_n\}$  and with a trivial kernel. Then there exists a rank one perturbation  $L^\circ$  of  $A^\circ$  with real spectrum such that both  $L^\circ$  and  $(L^\circ)^*$  have complete sets of eigenvectors, but  $L^\circ$  does not admit the spectral synthesis.*

As Marcus showed in [20], most classical sufficient conditions of completeness imply already that the operator admits the spectral synthesis.

**5.4. Removability of the spectrum.** In this subsection we address the following question, which, in a sense, is opposite to the completeness problem: *For which measures  $\mu = \sum_n \mu_n \delta_{t_n}$  does there exist a singular perturbation  $L$  of  $A$ , whose spectrum is empty (in other words, consists only of the point at infinity)?*

Thus, the problem is to describe those spectra  $\{t_n\}$  for which the spectrum of the perturbation is empty. Such spectra will be said to be *removable*. It is clear that the property to be removable or nonremovable depends only on  $\{t_n\}$ , but not on the choice of the masses  $\mu_n$ .

The removability criterion will be given in terms of entire functions of the so-called *Krein class*. We say that an entire function  $F$  is in the Krein class  $\mathcal{K}_1$ , if it has only real simple



zeros  $t_n$  and can be represented as

$$\frac{1}{F(z)} = q + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right), \quad \sum_n t_n^{-2} |c_n| < \infty,$$

where  $c_n = -1/F'(t_n)$  and  $q = 1/F(0)$ .

Then our main result in this section reads as follows:

**Theorem 5.7** ([4], Theorem 0.1). *Let  $t_n \in \mathbb{R}$  and  $|t_n| \rightarrow \infty$ ,  $|n| \rightarrow \infty$ . The following are equivalent:*

- (i) *The spectrum  $\{t_n\}$  is removable;*
- (ii) *There exists a function  $F \in \mathcal{K}_1$  whose zero set coincides with  $\{t_n\}$ .*

A somewhat unexpected and counterintuitive consequence of Theorem 5.7 is that adding a finite number of points to the spectrum helps it to become removable, while deleting a finite number of points can make it nonremovable. E.g., the spectrum  $t_n = n + 1/2$ ,  $n \in \mathbb{Z}$ , is removable, while  $t_n = n + 1/2$ ,  $n \in \mathbb{Z}$ ,  $n \neq 0$ , is not. Also, the spectrum  $\{n^2\}_{n \in \mathbb{N}}$  is removable, but  $\{n^2\}_{n \geq 2}$  is nonremovable.

In view of the relation between singular rank one perturbations (see (4.2)) and usual rank one perturbations of bounded self-adjoint operators, we have an immediate counterpart of Theorem 5.7 for Volterra rank one perturbation of compact operators.

**Theorem 5.8** ([4], Theorem 0.2). *Let  $s_n \in \mathbb{R}$ ,  $s_n \neq 0$ , and  $|s_n| \rightarrow 0$ ,  $|n| \rightarrow \infty$ , and let  $A^\circ$  be a compact self-adjoint operator with simple point spectrum  $\{s_n\}$  and with a trivial kernel. The following are equivalent:*

- (i) *There exists a rank one perturbation  $L^\circ$  of  $A^\circ$  which is a Volterra operator;*
- (ii) *The points  $t_n = s_n^{-1}$  form the zero set of some function  $F \in \mathcal{K}_1$ .*

We find it a bit surprising that such a natural question was not previously addressed. There is a vast literature on the subject, and many results (mostly due to Krein, Gohberg, and Matsaev) concern the relations between the real and the imaginary parts of a Volterra operator; see [13, Chapter III] and [12, Chapter IV] (especially, Section 10, where some partial results about Volterra operators with finite rank imaginary parts are obtained). Let us also mention a beautiful theorem of Livshits which says that a dissipative rank one perturbation of a self-adjoint operator is unitary equivalent to the integration operator (see [19] or [13, Chapter 1, Theorem 8.1]). Still, we were unable to find any result explicitly describing the compact self-adjoint operators whose rank one perturbation is a Volterra operator.

To conclude, we deduce the above-mentioned theorem of Livshits from the functional model of rank one perturbations<sup>2</sup>. Namely, we show the following:

**Theorem 5.9** (Livshits, [19]). *Let  $L^\circ = A^\circ + iB^\circ$  be a Volterra operator (in some Hilbert space  $H$ ) where both  $A^\circ$  and  $B^\circ$  are self-adjoint and  $B^\circ$  is of rank one. Then the point spectrum of  $A^\circ$  is given by  $s_n = c(n + 1/2)^{-1}$ ,  $n \in \mathbb{Z}$ , for some  $c > 0$ .*

From this, one can deduce that  $A^\circ$  is unitary equivalent to the integral operator (having the same spectrum)

$$(\tilde{A}f)(x) = i \int_0^{2\pi c} f(t) \operatorname{sign}(x - t) dt, \quad f \in L^2(0, 2\pi c),$$

while  $L^\circ$  is unitary equivalent to the integration operator  $(\tilde{L}f)(x) = 2i \int_0^{2\pi c} f(t) dt$ .

Since  $B^\circ = (B^\circ)^*$ , we have  $B^\circ x = (x, b^\circ)b^\circ$  for some  $b^\circ \in H$ . Passing to the unbounded inverses we obtain (after an obvious unitary equivalence) a singular rank one perturbation  $L = L(A, a, b, \varkappa)$  of the operator  $A$  of multiplication by the independent variable in some space  $L^2(\mu)$  where  $\mu = \sum_n \mu_n \delta_{t_n}$ ,  $t_n = s_n^{-1}$ . Moreover, in the case of the self-adjoint imaginary part, we may assume that  $\varkappa = -1$  and  $a = ib$ .

Applying the functional model from Section 4 we construct a de Branges space  $\mathcal{H}(E)$  and a function  $G$  as in (4.4)–(4.5). In our case we have

$$\begin{aligned} \frac{B_E(z)}{A_E(z)} &= \delta + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n, \\ \frac{G(z)}{A_E(z)} &= -1 + i \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n, \end{aligned}$$

whence  $G = -A_E + i(B_E - \delta A_E)$ .

Since  $L$  (and, thus, the model operator  $T$ ) is the inverse to a Volterra operator, the spectrum of  $T$  is the point at infinity. Thus,  $G$  has no zeros in  $\mathbb{C}$ . Also, by the results of [4], the function  $E$  is of Cartwright class and the same is true for  $G$ . We conclude that  $G(z) = \exp(i\pi cz)$  for some real  $c$ . Thus,

$$e^{i\pi cz} = -A_E(z) + i(B_E(z) - \delta A_E(z)).$$

The functions  $A_E$  and  $B_E$  are real on the real axis. Taking the real parts, we have  $A_E(z) = -\cos \pi cz$ , and so  $t_n = c^{-1}(n + 1/2)$ ,  $n \in \mathbb{Z}$ , as required.

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<sup>2</sup>It was N. Nikolski who attracted our attention to the Livshits theorem and suggested to deduce it using our methods.

## 6. OPEN PROBLEMS

Here we mention a few questions related to the above results. A very basic question, which remains open, is the following one.

**Problem 1.** *Is it true that any hereditarily complete system of exponentials in  $L^2(-\pi, \pi)$  is a linear summation basis (i.e., the corresponding Fourier series are all summable by a linear summation method)?*

We are able to construct systems  $\{K_\lambda\}_{\lambda \in \Lambda}$  of reproducing kernels in a Paley–Wiener space such that for some partition  $\Lambda = \Lambda_1 \cup \Lambda_2$  the orthogonal complement in  $\mathcal{PW}_\pi$  to the system  $\mathcal{F}_{\Lambda_1, \Lambda_2} = \{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$  is one-dimensional.

**Problem 2.** *How to characterize the vectors orthogonal to such systems?*

It is clear that given a partition  $\Lambda = \Lambda_1 \cup \Lambda_2$  such that  $\mathcal{F}_{\Lambda_1, \Lambda_2}$  has a nontrivial orthogonal complement, the orthogonal complement to the system  $\mathcal{F}_{\Lambda_1 \setminus \{\lambda_0\}, \Lambda_2 \cup \{\lambda_0\}}$  also will be nontrivial. Thus, we can always move a finite number of points from  $\Lambda_1$  to  $\Lambda_2$  and in the opposite direction.

**Problem 3.** *What is the structure of the set of those partitions  $\Lambda = \Lambda_1 \cup \Lambda_2$  for which the orthogonal complement to  $\mathcal{F}_{\Lambda_1, \Lambda_2}$  is nontrivial? Is this set in a sense "connected"?*

As we have seen in Example 3.4, there exist de Branges spaces  $\mathcal{H}(E)$  where any exact system of reproducing kernels is hereditarily complete. The proof in this example uses essentially that the spectrum  $\{t_n\}$  is lacunary. However, a de Branges space is uniquely (up to a natural isometry) defined by its spectral data  $t_n$  and  $\mu_n = 1/\varphi'(t_n)$ . Apart from the case of the Paley–Wiener spaces, in our examples of nonhereditarily complete systems only  $t_n$  were fixed.

**Problem 4.** *Describe spectral data  $(t_n, \mu_n)$  such that any exact system of the reproducing kernels in the corresponding de Branges space  $\mathcal{H}(E)$  is hereditarily complete. Is it true that for the spectrum  $\{t_n\}$  satisfying (3.3) and for any  $\mu_n$  with  $\sum_n \mu_n(t_n^2 + 1)^{-1} < \infty$  there exists a nonhereditarily complete system in  $\mathcal{H}(E)$ ?*

It should be mentioned that all our examples of nonhereditary completeness are constructed in the *reverse order*. Namely, we start with a vector  $h$  (in  $\mathcal{PW}_\pi$  or  $\mathcal{H}(E)$ ) with some special properties and then construct a sequence  $\Lambda$  such that  $h$  is orthogonal to some system  $\mathcal{F}_{\Lambda_1, \Lambda_2}$ . Therefore, we are not able to produce a nonhereditarily complete system of reproducing kernels in a de Branges space such that the orthogonal complement to the system  $\mathcal{F}_{\Lambda_1, \Lambda_2}$  is two-dimensional or infinite-dimensional. The existence of such examples

is an intriguing problem. Note that it follows from (the proof of) the theorem of Markus [20, Theorem 4.1] that if the system of eigenvectors is hereditarily complete up to a finite-dimensional defect, then the spectral synthesis holds up to a defect of the same dimension.

**Problem 5.** *Do there exist nonhereditarily complete systems of reproducing kernels in a de Branges space such that for some partition the orthogonal complement to  $\mathcal{F}_{\Lambda_1, \Lambda_2}$  is  $n$ -dimensional with  $n \geq 2$  or even infinite-dimensional?*<sup>3</sup>

Finally, though the spectral synthesis may fail even for exponential systems (reproducing kernels in the Paley–Wiener space), it may of interest to distinguish those systems for which it still holds. The generating function  $G$  may provide a natural language for this problem.

**Problem 6.** *What conditions on the generating function  $G$  of an exact system of reproducing kernels in  $\mathcal{H}(E)$  ensure that the system is hereditarily complete?*

Recently, this problem was considered by Gubreev and Tarasenko [14] who showed that in the case when  $|G/E|^2$  is a Muchenhaupt  $A_2$ -weight on  $\mathbb{R}$ , the corresponding system of kernels is hereditarily complete.

To conclude, it seems that hereditary completeness for systems of reproducing kernels in de Branges spaces and the related spectral synthesis problems (even for such simple class of linear operators as rank one perturbations of self-adjoint operators) remain a rich field where there is still much to explain and to explore.

## REFERENCES

- [1] A. Baranov, Yu. Belov, Systems of reproducing kernels and their biorthogonal: completeness or incompleteness? *Int. Math. Res. Notices* **22** (2011) 5076–5108.
- [2] A. Baranov, Y. Belov, A. Borichev, Hereditary completeness for systems of exponentials and reproducing kernels, arXiv:1112.5551, to appear in *Adv. Math.*
- [3] A. Baranov, D. Yakubovich, Completeness and spectral synthesis of nonselfadjoint one-dimensional perturbations of selfadjoint operators, Preprint, to appear on www.arxiv.org.
- [4] A. Baranov, D. Yakubovich, One-dimensional singular perturbations of unbounded self-adjoint operators with empty spectrum, Preprint.
- [5] Yu. Belov, T. Mengestie, K. Seip, Discrete Hilbert transforms on sparse sequences, *Proc. Lond. Math. Soc.* **103** (2011) 3, 73–105.
- [6] Yu. Belov, T. Mengestie, K. Seip, Unitary discrete Hilbert transforms, *J. Anal. Math.* **112** (2010) 383–395.

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<sup>3</sup> **Added in proof.** Recently the first three authors have constructed an example of a de Branges space with a nonhereditarily complete system  $\{K_\lambda\}_{\lambda \in \Lambda}$  such that, for some partition, the system  $\mathcal{F}_{\Lambda_1, \Lambda_2}$  has infinite defect.

- [7] L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice-Hall, Englewood Cliffs, 1968.
- [8] D. Deckard, C. Foiaş, C. Pearcy, Compact operators with root vectors that span, *Proc. Amer. Math. Soc.* **76** (1979) 1, 101–106.
- [9] L. Dovbysh, N. Nikolski, Two methods for avoiding hereditary completeness, *Zap. Nauchn. Semin. LOMI* **65** (1976) 183–188; English transl.: *J. Soviet Math.* **16** (1981) 3, 1175–1179.
- [10] L. Dovbysh, N. Nikolski, V. Sudakov, How good can a nonhereditary family be? *Zap. Nauchn. Semin. LOMI* **73** (1977) 52–69; English transl.: *J. Soviet Math.* **34** (1986) 6, 2050–2060.
- [11] E. Fricain, Complétude des noyaux reproduisants dans les espaces modèles, *Ann. Inst. Fourier (Grenoble)* **52** (2002) 2, 661–686.
- [12] I. Gohberg, M. Krein, *Introduction to the Theory of Linear Nonself-adjoint Operators*, Amer. Math. Soc., Providence, RI, 1969.
- [13] I. Gohberg, M. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Amer. Math. Soc., Providence, RI, 1970.
- [14] G. Gubreev, A. Tarasenko, Spectral decomposition of model operators in de Branges spaces, *Mat. Sb.* **201** (2010), 11, 41–76; Engl. transl.: *Sb. Math.* **201** (2010), 11, 1599–1634.
- [15] H. Hamburger, Über die Zerlegung des Hilbertschen Raumes durch vollstetige lineare Transformationen, *Math. Nachr.* **4** (1951) 56–69.
- [16] V. Kapustin, One-dimensional perturbations of singular unitary operators, *Zap. Nauchn. Semin. POMI* **232** (1996) 118–122; English transl.: *J. Math. Sci. (New York)* **92** (1998) 1, 3619–3621.
- [17] S. Khrushchev, N. Nikolski, A function model and some problems in spectral function theory, *Trudy Math. Inst. Steklov* **176** (1987) 97–210; English transl.: *Proc. Steklov Inst. Math.* **3** (1988) 101–214.
- [18] M. Krein, On perturbation determinants and the trace formula for unitary and self-adjoint operators, *Dokl. Akad. Nauk SSSR* **144** (1962) 268–271; English transl.: *Sov. Math. Dokl.* **3** (1962) 707–710.
- [19] M. Livshits, On spectral decomposition of linear non-self-adjoint operators, *Mat. Sb.*, **34(76)** (1954) 1, 145–199 (in Russian).
- [20] A. Markus, The problem of spectral synthesis for operators with point spectrum, *Izv. Akad. Nauk SSSR* **34** (1970) 3, 662–688; English transl.: *Math. USSR-Izv.* **4** (1970) 3, 670–696.
- [21] V. Matsaev, On a class of completely continuous operators, *Dokl. Akad. Nauk SSSR* **139** (1961) 3, 548–551; English transl.: *Soviet Math. Dokl.* **2** (1961) 972–975.
- [22] N. Nikolski, Complete extensions of Volterra operators, *Izv. Akad. Nauk SSSR* **33** (1969) 6, 1349–1355; English transl.: *Math. USSR-Izv.* **3** (1969) 6, 1271–1276.
- [23] N. Nikolski, Selected problems of weighted approximation and spectral analysis, *Trudy Mat. Inst. Steklov.* **120** (1974); English transl.: *Proc. Steklov Inst. Math.* **120** (1974).
- [24] N. Nikolski, The current state of the problem of spectral analysis-synthesis. I, *Operator theory in function spaces*, (Proc. School, Novosibirsk, 1975, 240–282, Izdat. Nauka, Sibirsk. Otdel., Novosibirsk, 1977; English transl.: *Amer. Math. Soc. Transl. Ser. 2* **124** (1984) 97–129.
- [25] N. Nikolski, *Treatise on the Shift Operator*, Springer-Verlag, Berlin, 1986.

- [26] N. Nikolski, *Operators, Functions, and Systems: an Easy Reading. Vol. 1–2*, Math. Surveys Monogr., Vol. 92–93, AMS, Providence, RI, 2002.
- [27] V. Ryzhov, Functional model of a class of non-self-adjoint extensions of symmetric operators, *Operator Theory: Advances and Applications*, Vol. 174, Birkhäuser, Basel, 2007, 117–158.
- [28] B. Solomyak, Complete extensions of compact operators with preservation of class, *Funkc. Anal. Pril.* **14** (1980) 3, 93–94; English transl.: *Functional Anal. Appl.* **14** (1980) 3, 244–245.
- [29] V. Vasyunin, Bases of eigensubspaces and nonclassical interpolation problems, *Funkc. Anal. Pril.* **9** (1975) 4, 65–66; English transl.: *Functional Anal. Appl.* **9** (1975) 4, 327–328.
- [30] J. Wermer, On invariant subspaces of normal operators, *Proc. Amer. Math. Soc.* **3** (1952) 2, 270–277.
- [31] R. Young, On complete biorthogonal system, *Proc. Amer. Math. Soc.* **83** (1981) 3, 537–540.

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